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BLOCKING OF THE FLOW OF A TWO-LAYER MIXING LIQUID AROUND AN OBSTACLE†

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Steady flow around a body when it is towed along the bottom of a channel in a thin homogeneous layer of a two-layer liquid which is initially at rest is investigated using a model of the flow of two-layer shallow water with mixing between the layers [1]. It is shown that, in the case of supercritical towing conditions, a local subcritical zone is formed ahead of the body and that the length of this zone depends on the height of the body.

When a stratified liquid flows under supercritical conditions over an uneven bottom, linear waves cannot propagate from an obstacle upwards through the liquid and the phenomenon of blocking, that is, the retardation of the fluid particles in a certain quite narrow flow domain, is associated with the substantially non-linear characteristics in the propagation of long waves. The structure of long plane waves in a two-layer liquid which are generated during the towing of a body has been investigated theoretically and experimentally [2–5]. The propagation of a non-linear wave upwards through the liquid reorganizes the flow. In turn, the velocity drift between the layers which arises is a source for the generation of short waves and of intermixing at the interface between the homogeneous layers. The concerted action of non-linear effects an intermixing can lead to a steady flow pattern during which a finite zone of partially blocked liquid is formed in front of the body. The steady, continuous solution, obtained in this paper, of the problem of the blocking of a flow accompanying the supercritical flow around the body reflects these characteristics of the flow of mixing liquids and considerably extends the range of applicability of the mathematical model.

1. Suppose a channel with a horizontal bottom is filled with a two-layer fluid of depth H which is at rest and, moreover, let the depth h_0 of the lower, heavier layer be small compared with H. The problem consists of describing the structure of the waves in the neighbourhood of an immersed body of height $\delta_{\max}$ which is towed along the bottom at a constant velocity D.

In the Boussinesq approximation $((\rho-\rho^+)/\rho^+ \ll 1)$, plane-parallel flow is characterized by two dimensionless parameters $\delta=\delta_{\max}/h_0$ and $\mathrm{Fr}=D/\sqrt{(bh_0)}$, where $b=(\rho^- -\rho^+)g/\rho^+$ is the buoyancy of the lower layer and ρ^+ and ρ^- are the densities of the upper and lower layers and g is the acceleration due to gravity.

In the case of non-mixing liquids when $h_0/H \ll 1$, the equations of motion are identical to those for a homogeneous shallow water with a "modified" gravitational acceleration $g'=b$. A diagram of the waves which are generated during a sudden motion of the body (Fr, δ), has been given in [2] for this case. In the case of a supercritical towing regime (Fr > 1), a steady, completely supercritical flow regime over the body is possible when $\delta<\delta_1(\mathrm{Fr})=\mathrm{Fr}^2/2+1= 3\mathrm{Fr}^{2/3}/2$ and there are no perturbations of the flow in front of or behind the body. On the other hand, when $\delta>\delta_2(\mathrm{Fr})$, an internal hydraulic jump can be formed which

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propagates at a velocity $D_1 \geq D$. At the same time, the flow which is continuous and steady with respect to the body changes into a subcritical flow ($|u_1 - D| < \sqrt{(bh_1)}$) in front of the body and a supercritical flow ($|u_2 - D| > \sqrt{(bh_2)}$) behind it. Here, h_i and u_i are the depth and velocity of the flow in front of ($i=1$) and behind ($i=2$) the body. The monotonic dependence $\delta = \delta_2(\mathrm{Fr})$ is found from the condition $D_1 = D$, and the relationship $\delta_2(\mathrm{Fr}) < \delta_1(\mathrm{Fr})$ is satisfied for $Fr > 1$ [1]. The possible flow pattern is therefore non-unique in the case of supercritical flow when $\delta_2(\mathrm{Fr}) < \delta < \delta_1(\mathrm{Fr})$. Experimentally, this non-uniqueness of the flow, which has been pointed out above, has been detected in a much narrower region [4].

The problem of the non-uniqueness of the steady flow is intimately associated with the problem of the choice of relationships in internal hydraulic jumps. Thus, if the law of conservation of total momentum in the lower layer, which is applied as in the case of a single liquid layer, is replaced by the condition that the flow is potential, then the "hysteresis" disappears but the total momentum is not conserved in a two-layer system. While there are different approaches to the derivation of relationships at discontinuities [6, 7], this problem cannot be solved within the framework of a model of a two-layer flow. The above-mentioned contradiction is removed in the more complete three-layer model [1] which takes account of such phenomena as intermixing and the generation of short waves at the interfaces of the homogeneous layers.

In the case of mixing liquids, the formulation of the problem on the blocking of the flow and the governing parameters remain the same as in the case of non-mixing liquids, but the structure of the waves, which are generated by the motion of a body, changes substantially. Because of the inhomogeneity of the equations, the solution no longer tends to a self-similar solution which makes the description of the asymptotic form more difficult. On the other hand, the role of steady flows in the neighbourhood of the body increases since, unlike in the case of non-mixing liquids, a wave which has left the body comes to a halt due to mixing in the case of a certain range of parameters. Such stationary solutions, which describe the effect of partial blocking of the flow during supercritical flow are fundamentally new and considerably extend the range of applicability of the equations of two-layer, shallow water with an irregular interface.

In the Boussinesq approximation, the system of equations for the plane-parallel flow of a thin homogeneous layer of a heavy liquid in a flooded space taking mixing and small scale motion into account has the form [1]

$$
\begin{aligned}
&(h+\eta/2)_t + (hu+\eta\upsilon/2)_x = 0\\
&(bh+\bar{b}\eta)_t + (bhu+\bar{b}\eta\upsilon)_x = 0\\
&u_t + (u^2/2 + bh + \bar{b}\eta)_x = 0\\
&(hu+\eta\upsilon)_t + (hu^2+\eta\upsilon^2+bh^2/2+\bar{b}\eta h+\bar{b}\eta^2/2)_x = 0\\
&(hu^2+\eta(\upsilon^2+q^2)+bh^2+2\bar{b}\eta h+\bar{b}\eta^2)_t +\\
&+(hu^3+\eta\upsilon(\upsilon^2+q^2)+2bh^2u+2\bar{b}(h+\eta)\eta\upsilon+2\bar{b}\eta hu)_x = 0\\
&\eta_t + (\eta\upsilon)_x = 2\sigma q \quad (\bar{b} = (\bar{\rho}-\rho^+)g/\rho^+
\end{aligned}
\tag{1.1}
$$

Here, h, η are depths, u, υ are the horizontal components of the mean velocity in the lower layer and the interlayer, $\bar{\rho}$ is the density in the interlayer and q is the velocity of the "large vortices" which are responsible for the drawing in of liquid from the homogeneous layers into the interlayer. The numerical value of the coefficient $\sigma = 0.15$ is determined from an analysis of the mixing in a homogeneous liquid and only has an effect on the ratio of the vertical and horizontal scales since the parameter σ can be eliminated from system (1.1) by an appropriate elongation of the independent variables. System (1.1) is obtained by adding the complete laws of conservation of mass, momentum and energy to the usual "shallow water" equations for the lower layer, and the corresponding quantities in the interlayer between the homogeneous layers are determined from these.

By virtue of (1.1), the process of drawing in a liquid is symmetric with respect to the centre of the interlayer and the relationship $\bar{b} = b/2$ holds until $h > 0$. When $h \equiv 0$, system (1.1) describes the evolution of a turbulent homogeneous layer [8]. In this case, the rate at which liquid is drawn into the interlayer is halved since nothing is drawn into the lower layer and the magnitude of $\bar{b}$ becomes variable.

System (1.1) is written in the form of laws of conservation and therefore determines both the continuous and discontinuous solutions. The relationships in the internal hydraulic jumps are uniquely derived from the conditions at the discontinuities. It has been shown [1, 8] that the system under consideration possesses a rich set of solutions of the travelling-wave type. Amongst these, there are continuous, soliton-like solutions and discontinuous solutions of the "jump-wave" and the "gentle wind" type. The stationary solutions describe the basic features of the flow in the problem of the outflow of a jet into a liquid of different density which is at rest and of the formation of a mixing layer and its transition into an immersed stream. In particular, the maximum flow rate of a liquid which is drawn into a mixing layer or an immersed stream can be determined within the framework of the model without invoking additional hypotheses.

The simultaneous use in system (1.1) of the laws of conservation of momentum and energy raised the problem of passing to the limit of a two-layer flow within the framework of this model since, in the limit when $\eta \equiv 0$, these laws contradict one another. Actually, in the case of a wave which propagates at a velocity $D > \sqrt{(bh_0)}(\eta_0 = 0)$ in a liquid which is at rest, it would appear that neither a continuous profile (the flow is supercritical with respect to the wave front) nor a discontinuous profile ($\eta = 0$ behind the wave by virtue of the laws of conservation) can exist, and the laws of conservation of momentum and energy cannot be satisfied simultaneously. However, the peculiarities of the behaviour of the characteristics of a three-layer flow and the fact that the drawing in process is taken into account in Eqs (1.1) enable one to resolve this paradox. In the following section it will be shown that, when $\mathrm{Fr} > 1$, a domain of blocked liquid of finite length is formed in front of the body and, in this domain, the flow is steady and subcritical with respect to the body around which the flow occurs. Moreover, when $\eta \to 0$, the solution does not contain singularities by virtue of the fact that the part of the liquid in the interlayer is accelerated up to a velocity which is close to the towing velocity D.

2. Solutions of system (1.1) when $\eta_0 = 0$, $u_0 = 0$ and $\mathrm{Fr} > 1$ (Fig. 1) which are stationary with respect to the body are considered. As was pointed out above, $\bar{b} = b/2$ in the interlayer and the conservation laws (1.1) take the form

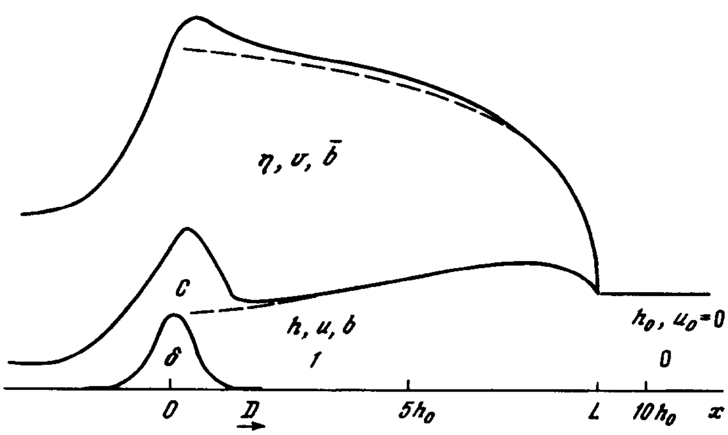

Fig. 1.

$$\begin{aligned}
&hu+\eta\upsilon/2-D(h+\eta/2)=-Dh_0\\
&u^2/2+b\eta/2+bh-Du=bh_0=J^-\\
&hu^2+\eta\upsilon^2-D(hu+\eta\upsilon)+b(h^2+h\eta+\eta^2/2)/2=bh_0^2/2\\
&\eta(\upsilon-D)(\upsilon^2+q^2)+h(u-D)u^2+b\eta hu+b(h+\eta)\eta\upsilon+2bh^2u-\\
&-bD(\eta^2/2+h\eta+h^2)=-bDh_0^2
\end{aligned} \tag{2.1}$$

By virtue of (2.1), all the required quantities can be expressed as functions of a single variable u, for example. The dependence of $Q=\eta(D-\upsilon)$ on u is shown in Fig. 2 for $\mathrm{Fr}=2$. Along branch B, when $u\to 0$, we have $\eta\to 0$, $h\to h_0$, $Q\to 0$. In the case of the function $\upsilon=\upsilon(u)$ the limit $\upsilon(0)=D$ can be found by differentiating the first three equations of (2.1) and taking the limit as $u\to 0$ in the resulting relationships. Next, in order to show that the limit $q^2(0)=D^2$ exists, it is necessary to reveal a singularity in the energy equation by differentiating relationships (2.1) twice. States in branch A are not considered since the function $q^{(2)}(u)$ is negative on this branch.

Only points for which $q^2(u)>0$ (the solid line in Fig. 2) have a physical meaning on branch B. A stationary solution can therefore take values Q only in the interval $(0, Q_0)$. The dependence of the required quantities on the variable $\xi=x-Dt$ is found from the equation

$$dQ(u)/d\xi=2\sigma q(u) \tag{2.2}$$

the solution of which, when account is taken of (2.1), can be obtained in quadratures. In particular, the dependence of the length of the stationary blocked zone L on the amount of liquid Q which is drawn into the interlayer follows from (2.2).

The motion of the obstacle completely determines the steady flow upwards and downstream in the case when the flow is supercritical behind its crest and small perturbations in the flow do not reach the segment 0–1 (Fig. 1). This is possible if the flow is critical on the crest (state c) and subcritical in the segment 0–1.

It has been noted [1] that the characteristics of system (1.1) consist of the characteristics of two-layer shallow water [9] and the multiple characteristic $dx/dt=\upsilon$. The condition for the criticality of state c is therefore determined by the vanishing of the characteristic determinant $\Delta(\lambda)$

$$\Delta(\lambda)=\left(\frac{(u-\lambda)^2}{bh}-1\right)\left(\frac{(\upsilon-\lambda)^2}{b\eta/2}-1\right)-\frac{1}{2}=0 \tag{2.3}$$

when $\lambda=D$, $u=u_c$, $\upsilon=\upsilon_c$, $h=h_c$, $\eta=\eta_c$.

In the case of a relatively small length of the body which is blocked, it is possible to neglect the process in which fluid is drawn in directly above it. Then, the closed relationships, which link state c on the crest and state 1 immediately ahead of the body, have the form

$$\begin{aligned}
&\eta_c(D-\upsilon_c)=\eta_1(D-\upsilon_1)=Q_1\\
&h_c(D-u_c)=h_1(D-u_1)=Dh_0-Q_1/2\\
&u_c^2/2-Du_c+b\eta_c/2+bh_c+b\delta=J^-\\
&\upsilon_c^2/2-D\upsilon_c+b(\eta_c+h_c+\delta)/2=\upsilon_1^2/2-D\upsilon_1+b(\eta_1+h_1)/2=J_1^+\\
&q_c^2=q_1^2
\end{aligned} \tag{2.4}$$

In the (u, Q) plane, the dependence $Q(u)$, which is represented by curve A in Fig. 2, corresponds to a supercritical flow $(\lambda_{\max}<D)$ while curve B corresponds to a subcritical flow $(\Delta(D)<0)$.

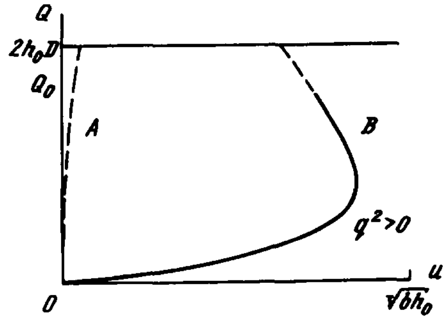

Fig. 2.

Here, λ_{max} is the greatest root of the equation $\Delta(\lambda) = 0$.

Hence, for any value of Q_1 in the range $(0, Q_0)$, a stationary subcritical blocked zone of finite length may be constructed in front of the body. Within this zone, Q varies from 0 to Q_1. The dependence $\delta = \delta(Q_1)$ is then uniquely found from relationships (2.3) and (2.4). Naturally, $\delta \to 0$ when $Q_1 \to 0$, and a whole domain adjoining the abscissa and corresponding to steady flow conditions arises in the (Fr, δ)-plane when $Fr > 1$. At sufficiently high towing velocities $(Fr > 2.1)$ and body height δ, the interlayer reaches the bottom (h_1 when $Q_1 = 2Dh_0$). An analysis of the running waves in the case of a completely mixed lower layer has been carried out in [8].

The stationary solution obtained in the problem of the blocking of the flow of a two-layer mixing liquid during supercritical flow around a body has a number of interesting properties. It provides an example of a self-consistent, continuous solution of a problem concerning a local zone of subcritical flow which does not have singularities but, at the same time, possesses a fine internal structure. In particular, the length of the zone where the flow is blocked is exceedingly sensitive to the height of the body and to the towing velocity. The stationary solution which has been constructed can therefore serve as a good test in the case of non-stationary calculations.

The numerical solution of an unsteady problem on the motion of an obstacle with a constant velocity $(Fr = 2)$ at long times is shown by the solid line in Fig. 1 while the corresponding stationary solution is shown by the dashed line. They are only observed to diverge from one another in the neighbourhood of the body. This is due to the fact that the finiteness of the length of the body has been taken into account in the non-stationary calculation. Calculations show that, in the case of towing parameters which correspond to steady flow, this regime is attained after a long time but the non-stationary segment of the acceleration of the body has a substantial effect on the intermediate picture of the flow and the phenomenon of "hysteresis", that is, a dependence of the flow picture on the method of accelerating the body, was also observed in a numerical experiment at times which are characteristic of experiments [4].

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